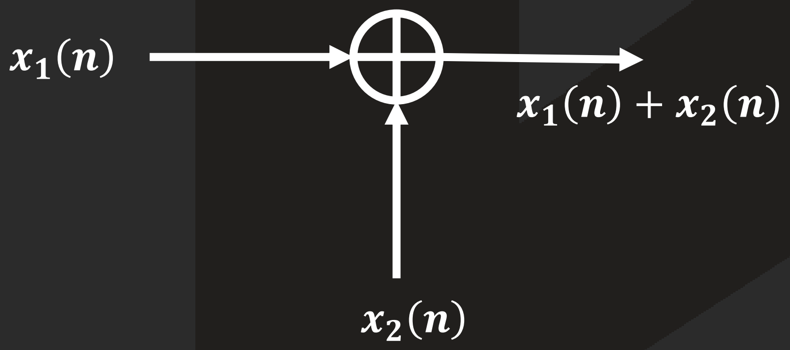
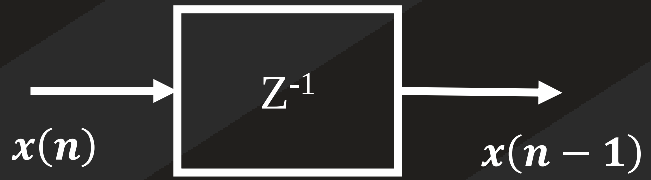
Realisation of discrete time system

- A digital filter transfer function can be realized in a variety ways
  - Realization of FIR filters
  - Realization of IIR filters
- Basic elements required for implementation of an LTI digital system are adder , multiplier and memory for storing elements
- In digital implementation the delay operation can be implemented by providing a storage register by each unit delay is required



Adder

Multiplier



Unit delay

FIR filter realization

- Mainly four types of realisation are there
  1. Direct form realisation
  2. Cascade form realisation
  3. Linear phase realisation
  4. Lattice structure realisation

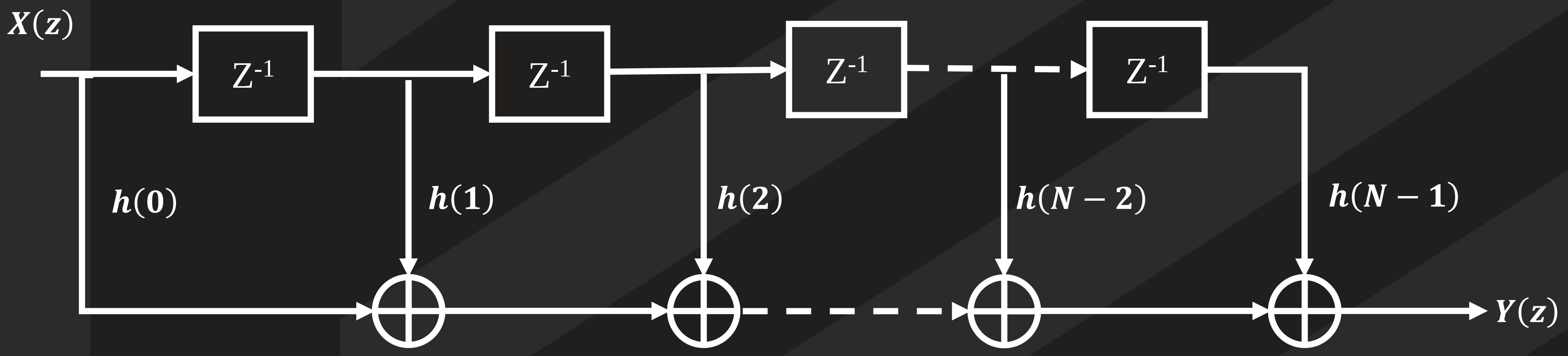
Direct form realization

The direct form of FIR may be obtained by using the equation of linear convolution

$$y(k) = \sum_{k=0}^{N-1} h(k)x(n-k) \qquad = h(0)x(n) + h(1)x(n-1) + \cdots + h(N-1)x(n-N-1)$$

Taking Z-transform

$$Y(z) = h(0)X(z) + h(1)X(z)z^{-1} + \cdots + h(N-1)X(z)z^{-(N-1)}$$

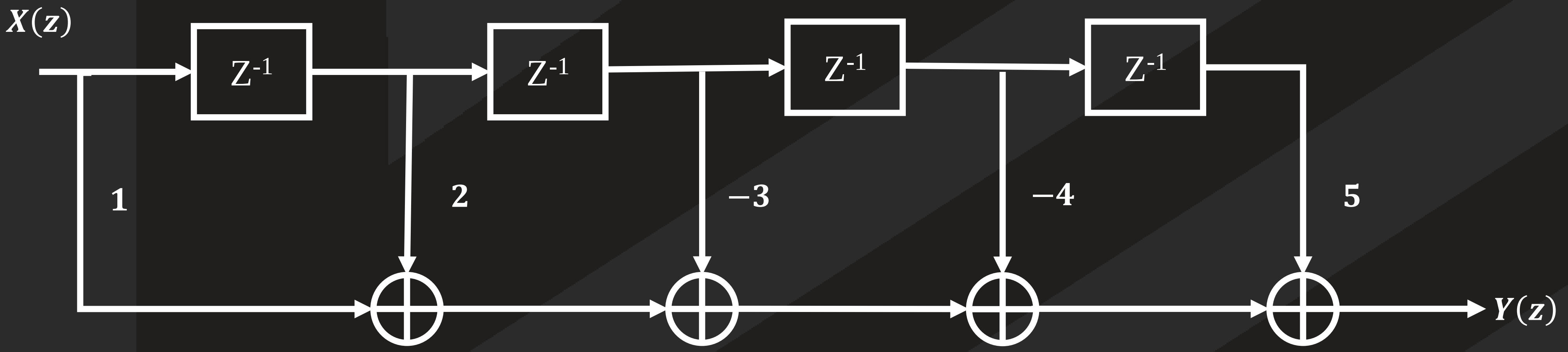


Q) *Determine the direct form realization of system function*
$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$$

$$Y(z) = X(z)[1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}]$$



Cascade form realization

Q) *Obtain cascade form realization of system function* $H(z) = (1 + 2z^{-1} - z^{-2})(1 + z^{-1} - z^{-2})$

$$H(z) = H_1(z)H_2(z)$$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)}$$

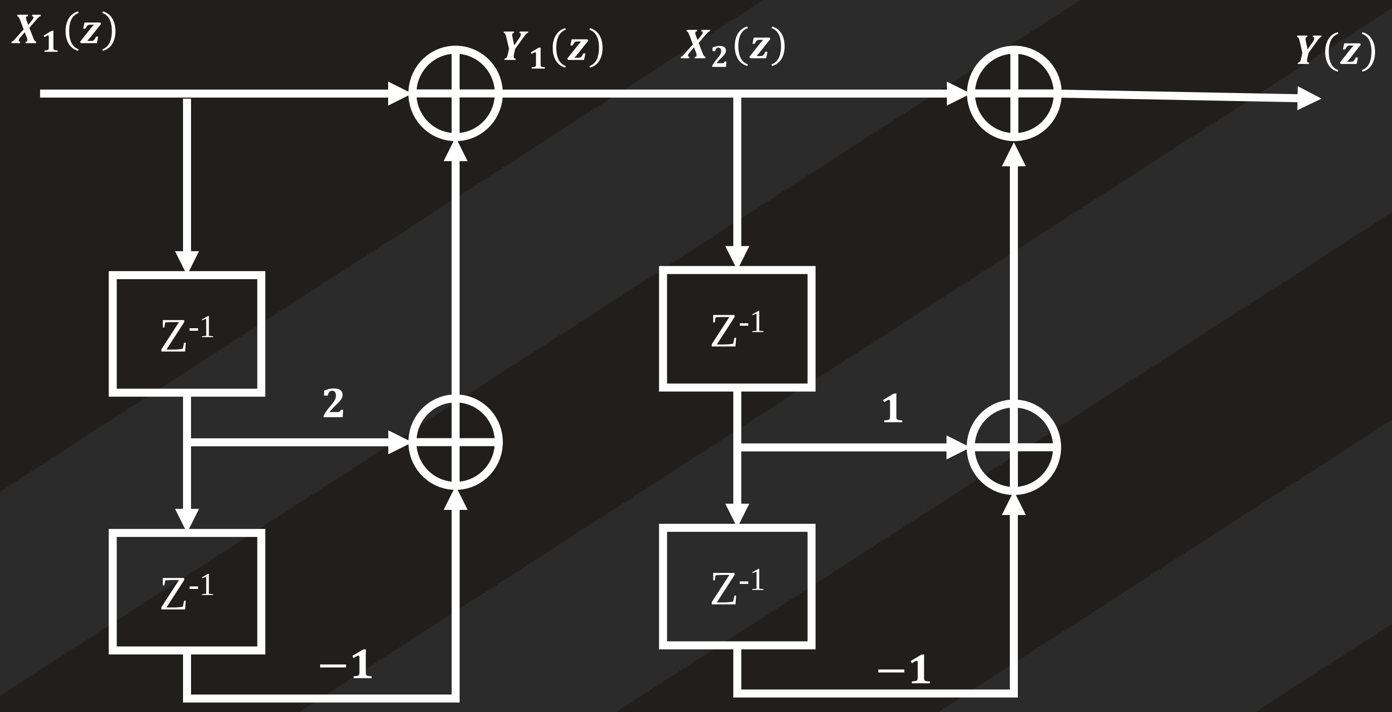
$$H_2(z) = \frac{Y_2(z)}{X_2(z)}$$

$$Y_1(z) = X_1(z)[1 + 2z^{-1} - z^{-2}]$$

$$Y_1(z) = X_1(z) + 2X(z)z^{-1} - X(z)z^{-2}$$

$$Y_2(z) = X_2(z)[1 + z^{-1} - z^{-2}]$$

$$Y_2(z) = X_2(z) + X_2(z)z^{-1} - X_2(z)z^{-2}$$



Cascade form realization

Q) Obtain cascade form realization of system function $H(z) = 1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$

$$H(z) = 1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$$

$$= \frac{z^3}{z^3}\left[1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}\right]$$

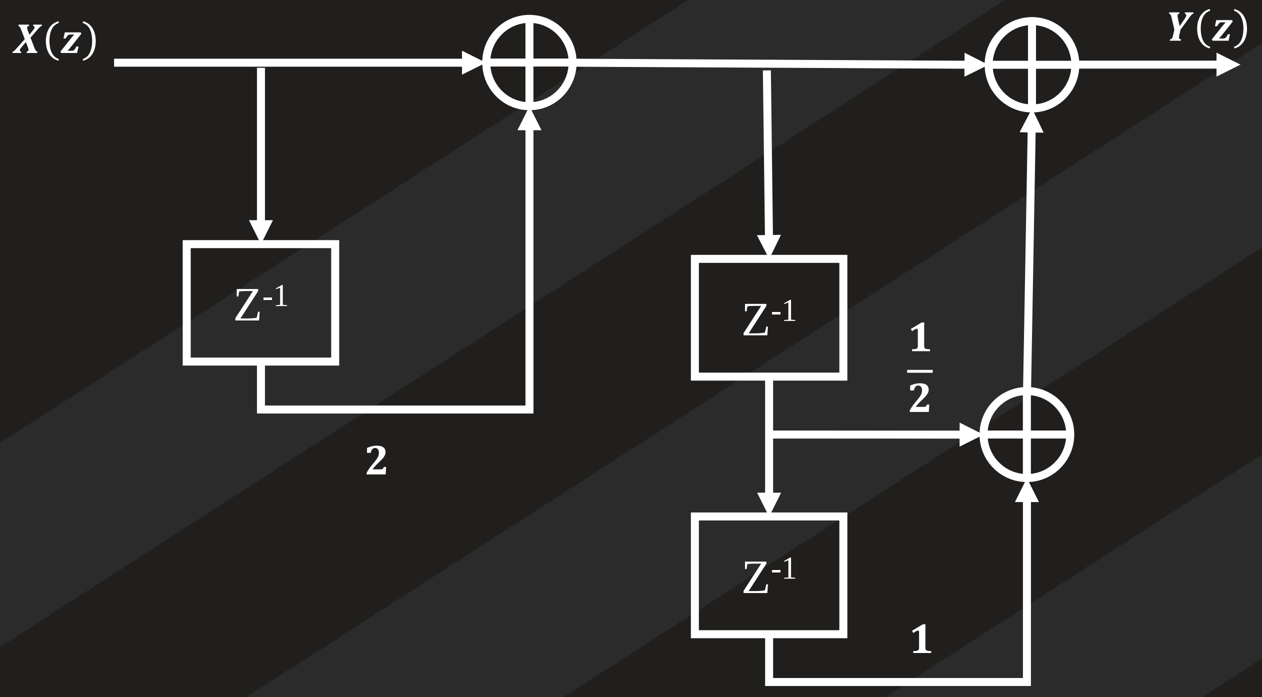
$$= \frac{1}{z^3}\left[z^3 + \frac{5}{2}z^2 + 2z^1 + 2\right]$$

The term inside the bracket equal to zero when z=-2

So the first term can be (z+2)

$$= \frac{1}{z^3}\left[(z+2)\left(z^2 + \frac{1}{2}z + 1\right)\right]$$

$$= \frac{1}{z^3}\left[z(1 + 2z^{-1})z^2\left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\right]$$



Linear phase realization

Q) Realise the system function $H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$

$h(0)$ $h(1)$ $\boldsymbol{h(2)}$ $\boldsymbol{h(3)}$ $\boldsymbol{h(4)}$ $\boldsymbol{h(5)}$ $\boldsymbol{h(6)}$

For a linear phase FIR filter $h(n) = h(N-1-n)$

$N = 7$

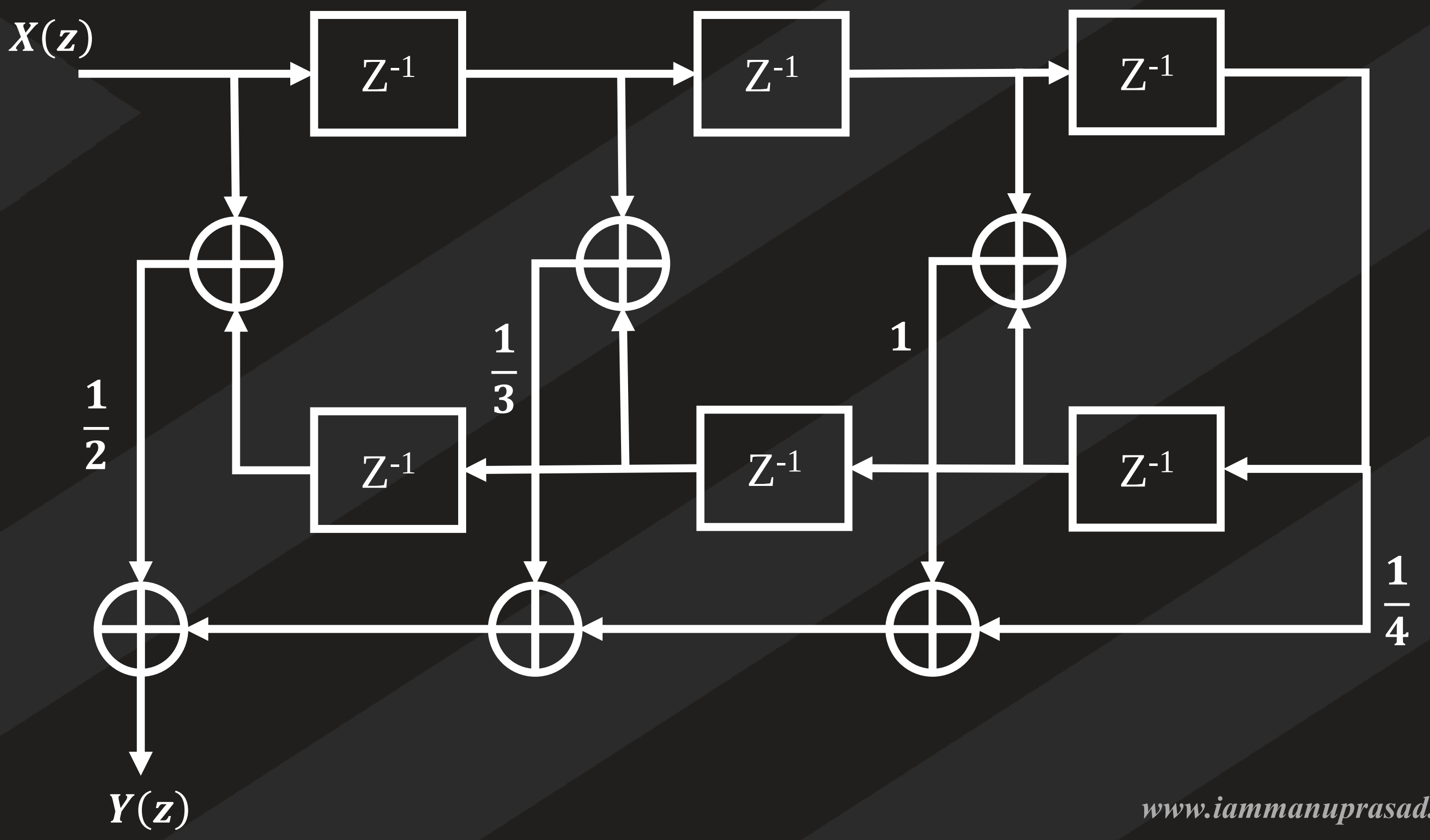
$h(0) = h(7-1-0) = h(6) = \frac{1}{2}$

$h(1) = h(7-1-1) = h(5) = \frac{1}{3}$

$h(2) = h(7-1-2) = h(4) = 1$

$h(3) = h(7-1-3) = h(3) = \frac{1}{4}$

$$H(z) = \frac{1}{2}(1 + z^{-6}) + \frac{1}{3}(z^{-1} + z^{-5}) + 1.(z^{-2} + z^{-4}) + \frac{1}{4}(z^{-3})$$



Linear phase realization

Q) Realise the system function using minimum number of multiplier

$$H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + z^{-5}$$

$h(0)$ $h(1)$ $h(2)$ $h(3)$ $h(4)$ $h(5)$

For a linear phase FIR filter $h(n) = h(N-1-n)$

$N = 6$

$h(0) = h(6-1-0) = h(5) = 1$

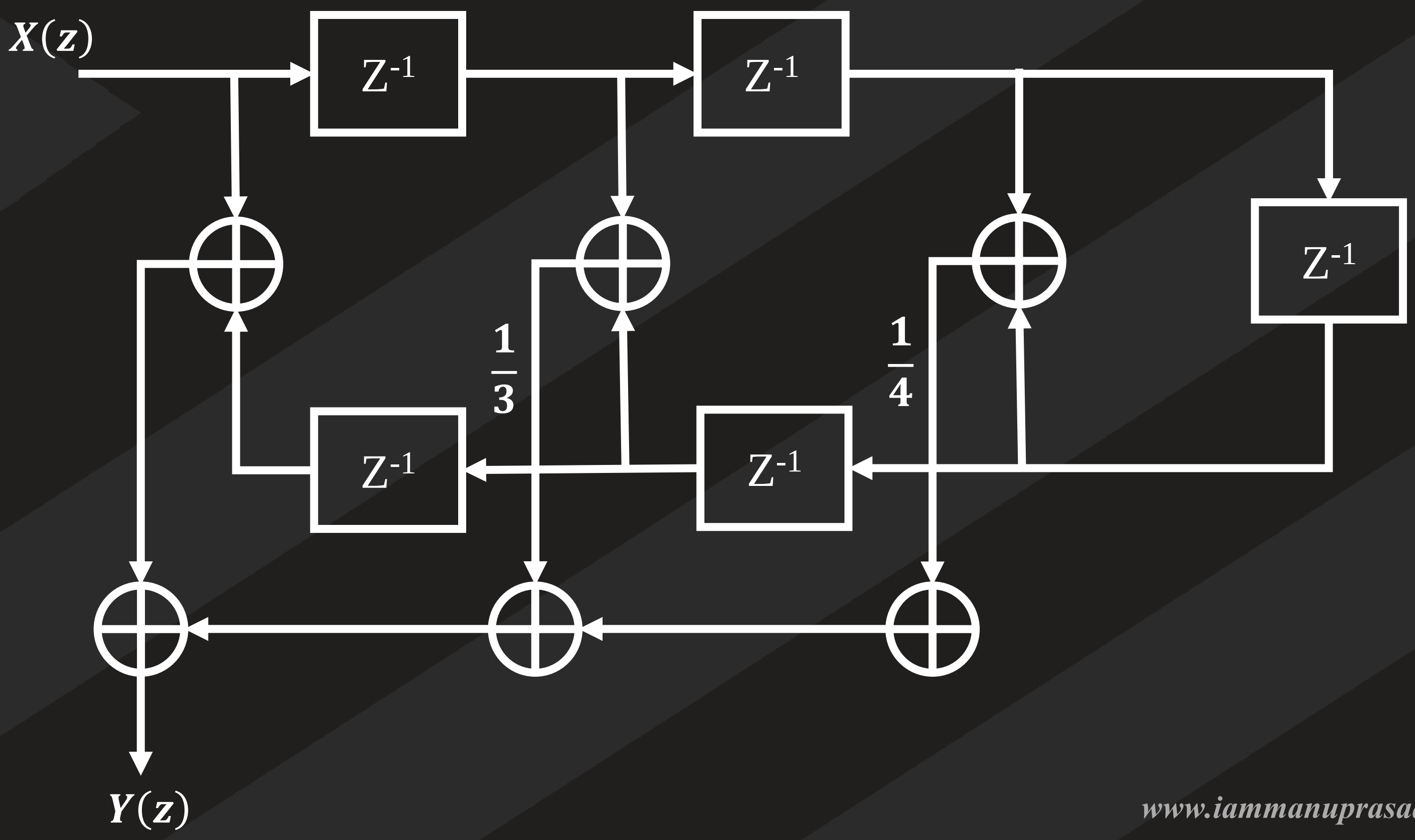
$h(1) = h(6-1-1) = h(4) = \frac{1}{3}$

$h(2) = h(6-1-2) = h(3) = \frac{1}{4}$

$H(z)$

$= 1(1 + z^{-5}) + \frac{1}{3}(z^{-1} + z^{-4})$

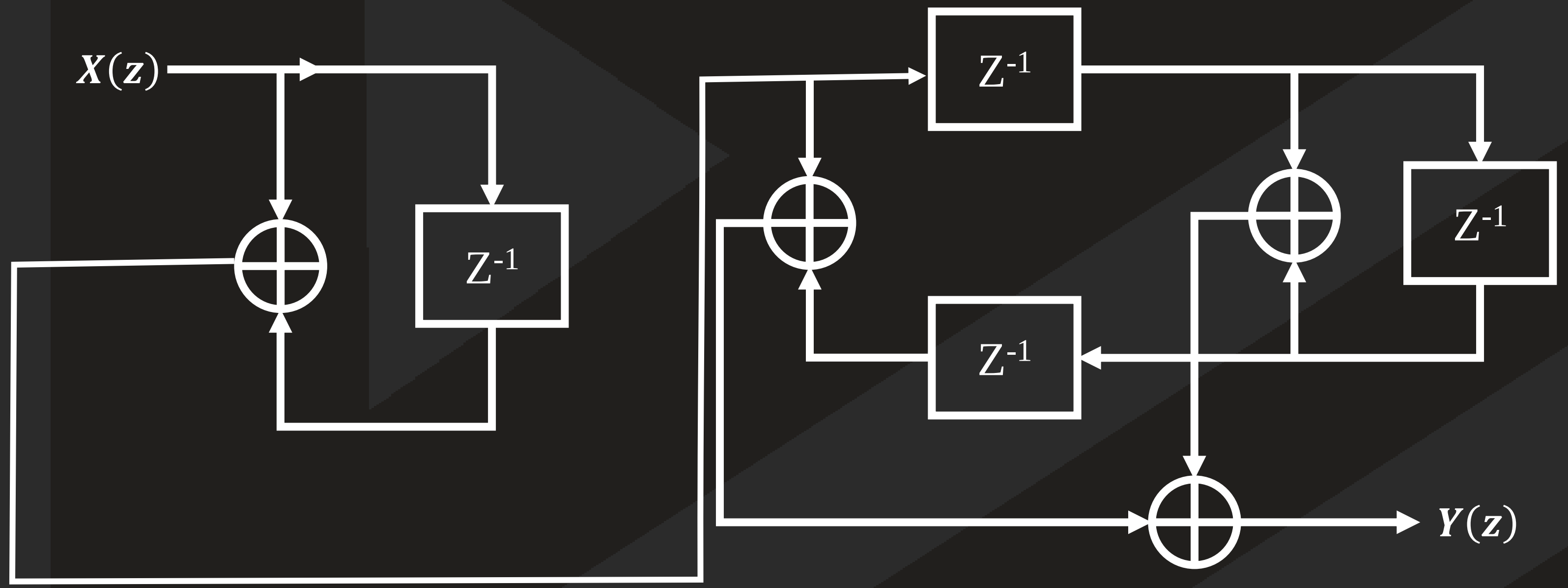
$+ \frac{1}{4}(z^{-2} + z^{-3})$



Linear phase realization

Q) Realise the system function using minimum number of multiplier

$$H(z) = (1 + z^{-1})\left(1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}\right)$$



Conversion of lattice coefficient to direct form filter coefficient

The general equation

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

The equations to convert filter coefficients to direct form FIR filter coefficients are

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = K_m$$

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k)$$

Q) Consider an FIR filter lattice filter with coefficient $K_1 = ½$,$K_2 = 1/3$, $K_3 = ¼$. Determine the FIR filter coefficients for direct form structure?

Solution

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

Number of stages (m) = 3

$$y(n) = x(n) + \sum_{k=1}^{3} \alpha_m(k)x(n-k) \qquad = x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3)$$

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = K_m$$

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k)$$

For m=3

$$\alpha_3(0) = 1$$

$$\alpha_3(3) = \frac{1}{4}$$

and k=2

$$\alpha_3(2) = \alpha_2(2) + \alpha_3(3)\alpha_2(1)$$

$$= \frac{1}{3} + \frac{1}{4}\alpha_2(1) \qquad = \frac{1}{2}$$

For m=2

$$\alpha_2(0) = 1$$

$$\alpha_2(2) = \frac{1}{3}$$

and k=1

$$\alpha_2(1) = \alpha_1(1) + \alpha_2(2)\alpha_1(1)$$

$$= \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} \qquad = \frac{2}{3}$$

For m=1

$$\alpha_1(0) = 1$$

$$\alpha_1(1) = \frac{1}{2}$$

For m=3 and k=1

$$\alpha_3(1) = \alpha_2(1) + \alpha_3(3)\alpha_2(2)$$

$$= \frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3} \qquad = \frac{3}{4}$$

Conversion of direct form FIR filter coefficient to lattice coefficient

The general equation

$$\boxed{y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)}$$

The equations to convert filter coefficients to lattice form FIR filter coefficients are

$$\alpha_{m-1}(0) = 1$$

$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \quad for\ 1 \le k \le m-1$$

A FIR filter is given by the difference equation $y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$ *Determine its lattice form*

Solution

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

$$y(n) = x(n) + \sum_{k=1}^{3} \alpha_m(k)x(n-k)$$

$$= x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3)$$

$$y(n) = 2\left[x(n) + \frac{2}{5}x(n-1) + \frac{3}{4}x(n-2) + \frac{1}{3}x(n-3)\right]$$

Comparing the two equations we get

$$\alpha_{m-1}(0) = 1$$

$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \quad for\ 1 \le k \le m-1$$

$$\alpha_3(1) = \frac{2}{5} \qquad \alpha_3(2) = \frac{3}{4} \qquad \alpha_3(3) = \frac{1}{3} = k_3$$

$$\alpha_2(0) = 1$$

$\alpha_3(1) = \frac{2}{5}$ $\alpha_3(2) = \frac{3}{4}$ $\alpha_3(3) = \frac{1}{3} = k_3$ $\alpha_2(0) = 1$

For m=3, k=1

$$\alpha_2(1) = \frac{\alpha_3(1) - \alpha_3(3)\alpha_3(2)}{1 - \alpha_3^2(3)} = \frac{\frac{2}{5} - \frac{1}{3}\cdot\frac{3}{4}}{1 - \left(\frac{1}{3}\right)^2} = 0.1687$$

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

For m=3, k=2

$$k_2 = \alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\alpha_3(1)}{1 - \alpha_2^2(3)} = \frac{\frac{3}{4} - \frac{1}{3}\cdot\frac{2}{5}}{1 - \left(\frac{1}{3}\right)^2} = 0.6937$$

For m=2, k=1

$$k_1 = \alpha_1(1) = \frac{\alpha_2(1) - \alpha_2(2)\alpha_2(1)}{1 - \alpha_2^2(2)} = \frac{0.1687 - (0.6937)0.1687}{1 - (0.6937)^2}$$

$$= 0.0996$$

$$\alpha_{m-1}(0) = 1$$

$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \quad for\ 1 \le k \le m-1$$

IIR filter Realisation

- Mainly four types of realisation are there
  1. Direct form I realisation
  2. Direct form II realisation
  3. Cascade form realisation
  4. Parallel form realisation

Direct form I realization

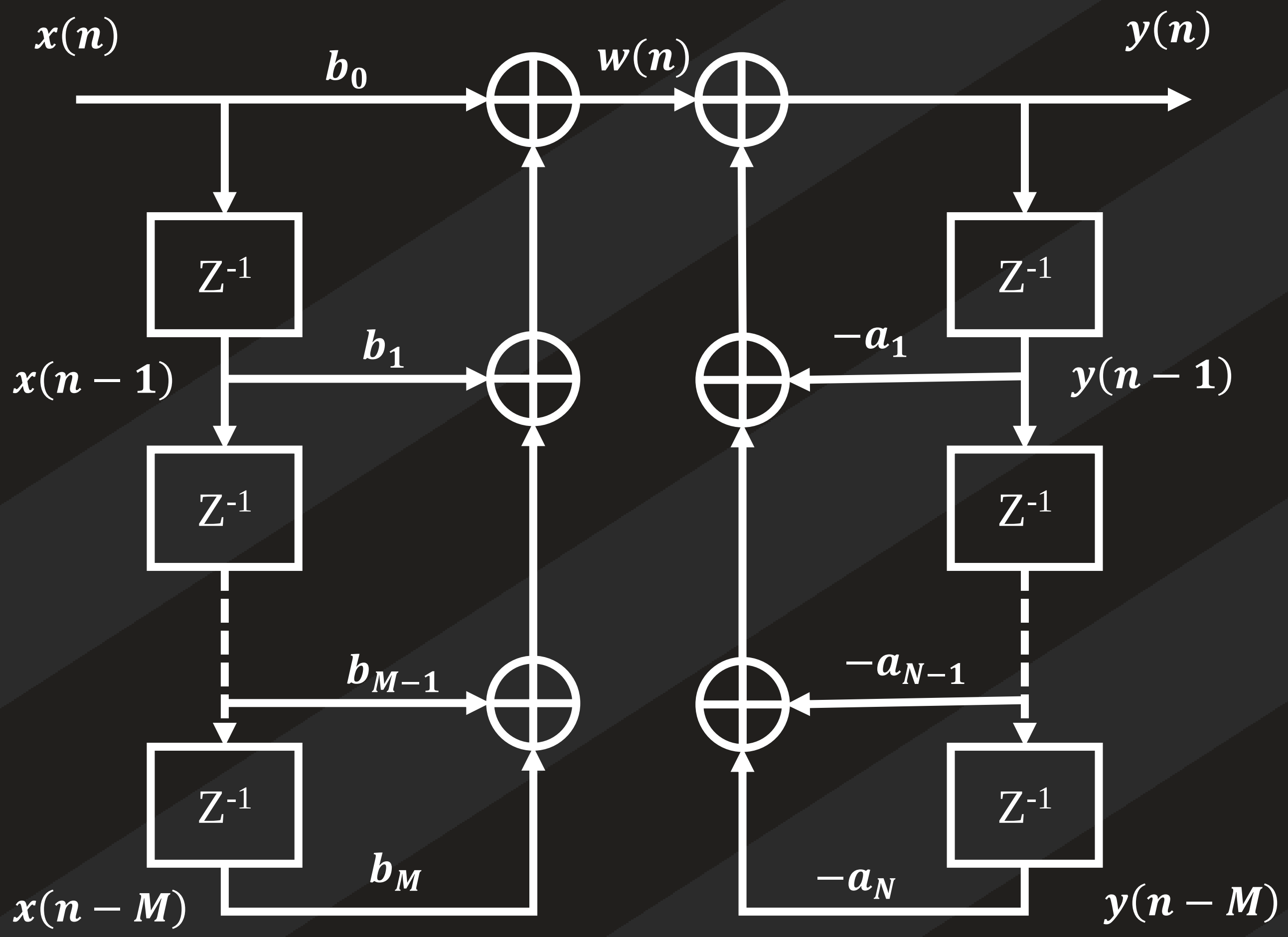
Let us consider an IIR system described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$= -a_1 y(n-1) - a_2 y(n-2) \ldots - a_N y(n-N) + w(n)$$

Where

$$w(n) = b_0 x(n) + b_1 x(n-N) + \cdots + b_M x(n-M)$$



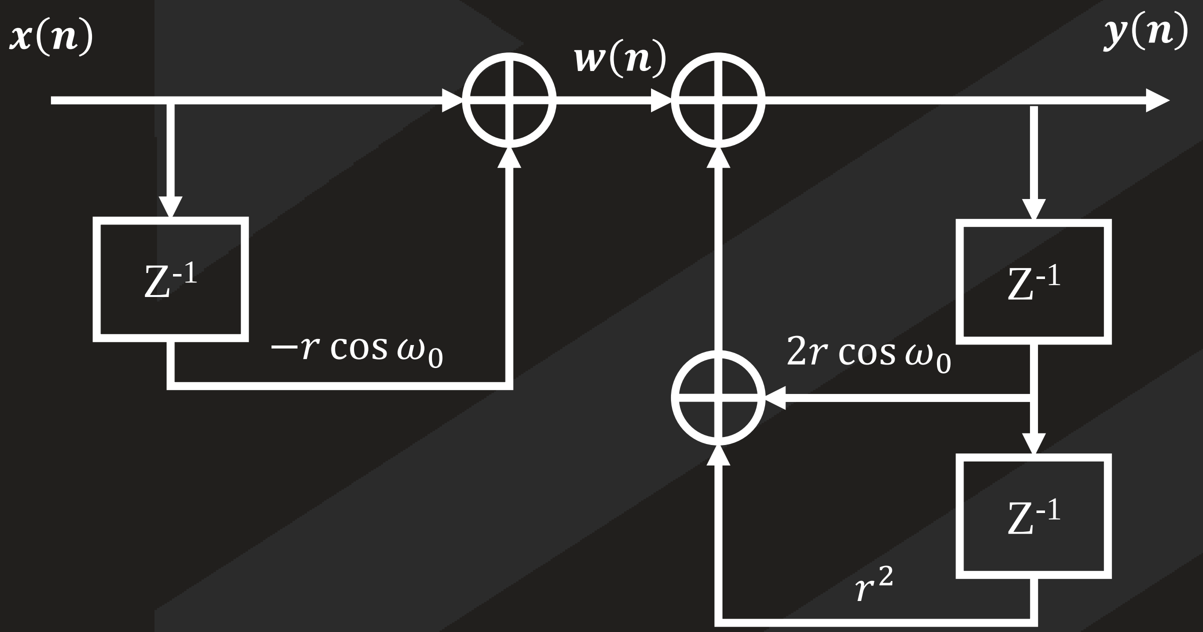
Direct form I realization

Q) Realise the second order digital filter

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

Solution

$$w(n) = x(n) - r\cos\omega_0\, x(n-1) \qquad y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + w(n)$$



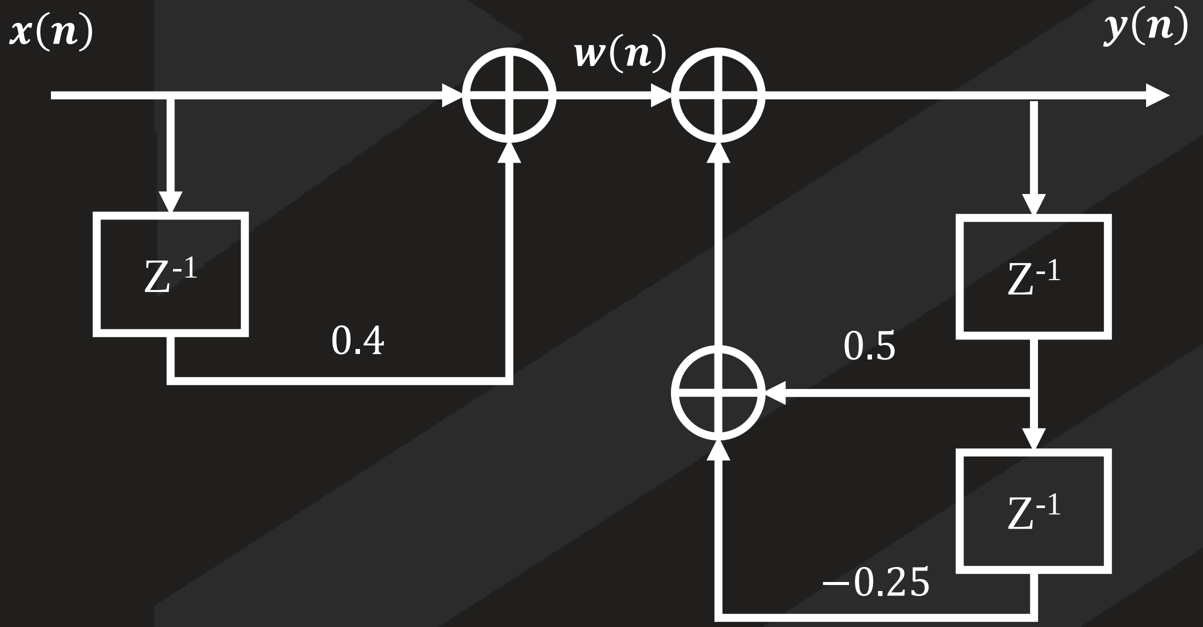
Direct form I realization

Q) Obtain the direct form I realisation for the system described by difference equation

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$$

Solution

$$w(n) = x(n) + 0.4x(n-1) \qquad y(n) = 0.5y(n-1) - 0.25y(n-2) + w(n)$$



Direct form II realization

Let us consider an IIR system described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

The system function can be represented as

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Let

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + \cdots + b_M z^{-M} W(z)$$

where

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) \ldots + a_N z^{-N} W(z) = X(z)$$

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \ldots - a_N z^{-N} W(z)$$

Taking inverse z transform we get

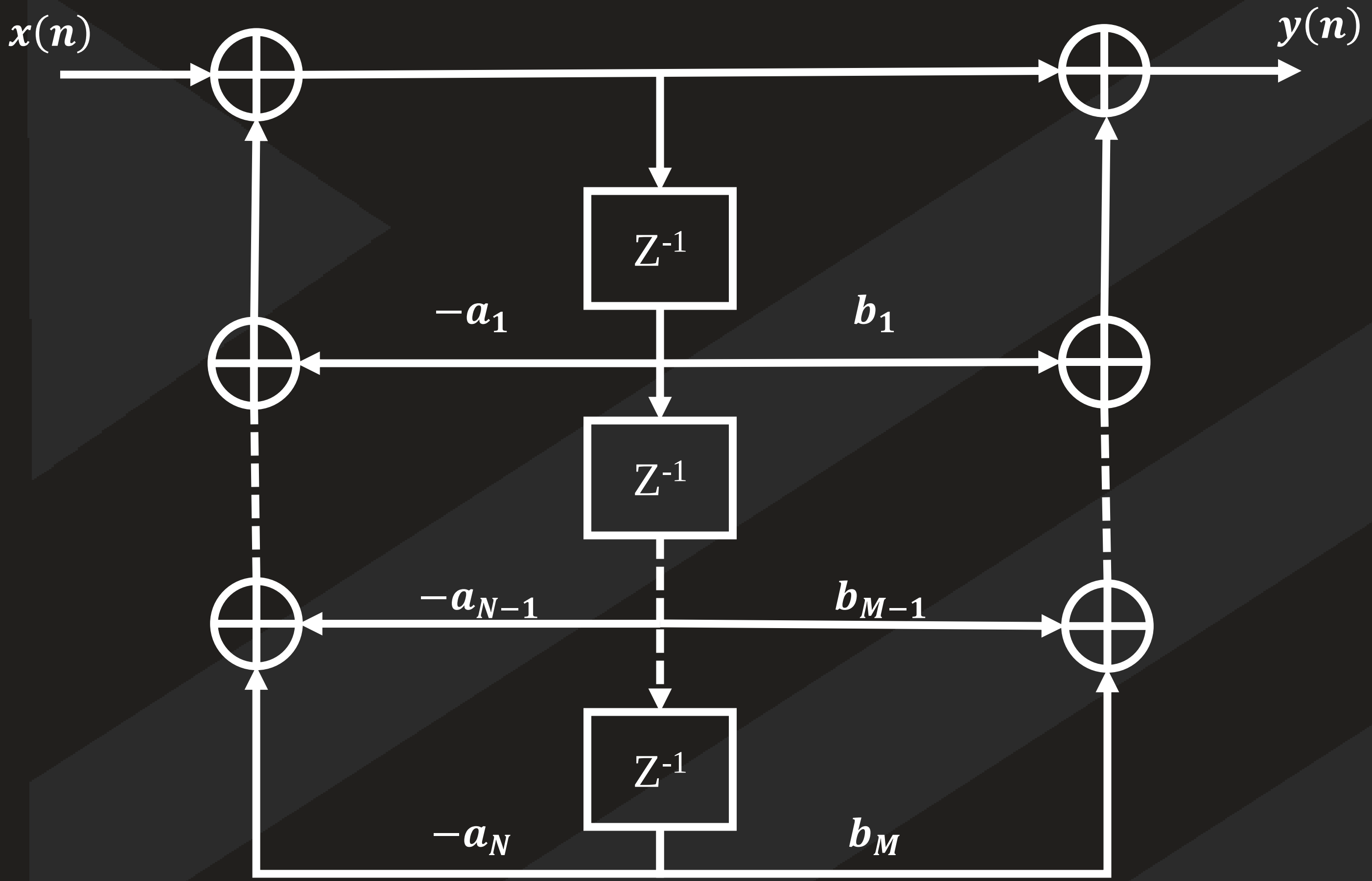
$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) \ldots - a_N w(n-N)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + \cdots + b_M w(n-M)$$

Direct form II realization

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) \ldots - a_N w(n-N)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + \cdots + b_M w(n-M)$$



Direct form II realization

Q) Realise the second order digital filter using direct form II

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

Solution

Taking Z - Transform

$$Y(z) = 2r\cos\omega_0\, Y(z)z^{-1} - r^2 Y(z)z^{-2} + X(z) - r\cos\omega_0\, X(z)z^{-1}$$

$$Y(z)[1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}] = X(z)[1 - r\cos\omega_0\, z^{-1}]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - r\cos\omega_0\, z^{-1}}{1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 1 - r\cos\omega_0\, z^{-1}$$

$$Y(z) = W(z)[1 - r\cos\omega_0\, z^{-1}]$$

Inverse transform

$$y(n) = w(n) - r\cos\omega_0\, w(n-1)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}}$$

$$W(z) - 2r\cos\omega_0\, W(z)z^{-1} + r^2 W(z)z^{-2} = X(z)$$

$$W(z) = X(z) + 2r\cos\omega_0\, z^{-1}W(z) - r^2 z^{-2} W(z)$$

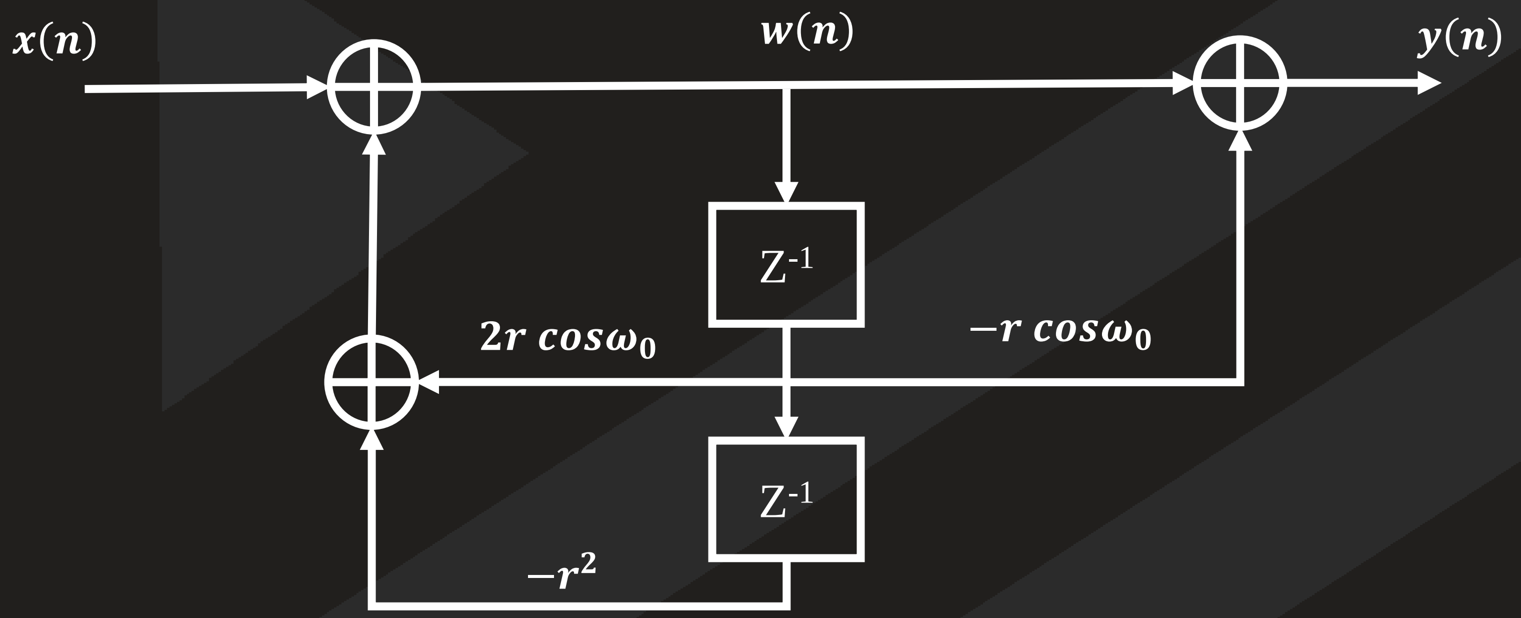
Inverse transform

$$w(n) = x(n) + 2r\cos\omega_0\, x(n-1) - r^2 x(n-2)$$

Direct form II realization

$$y(n) = w(n) - r\cos\omega_0\, w(n-1)z^{-1}$$

$$w(n) = x(n) + 2r\cos\omega_0\, x(n-1) - r^2 x(n-2)$$



Direct form II realization

Q) Determine the direct form II realisation for the following system

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-1)$$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

Solution

Taking Z - Transform

$$Y(z) = -0.1Y(z)z^{-1} + 0.72Y(z)z^{-2} + 0.7X(z) - 0.252X(z)z^{-1}$$

$$Y(z)[1 + 0.1z^{-1} - 0.72z^{-2}] = X(z)[0.7 - 0.252z^{-1}]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-1}$$

$$Y(z) = W(z)[0.7 - 0.252z^{-1}]$$

Inverse transform

$$y(n) = 0.7w(n) - 0.252w(n-1)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$W(z) + 0.1W(z)z^{-1} - 0.72W(z)z^{-2} = X(z)$$

$$W(z) = X(z) - 0.1W(z)z^{-1} + 0.72W(z)z^{-2}$$

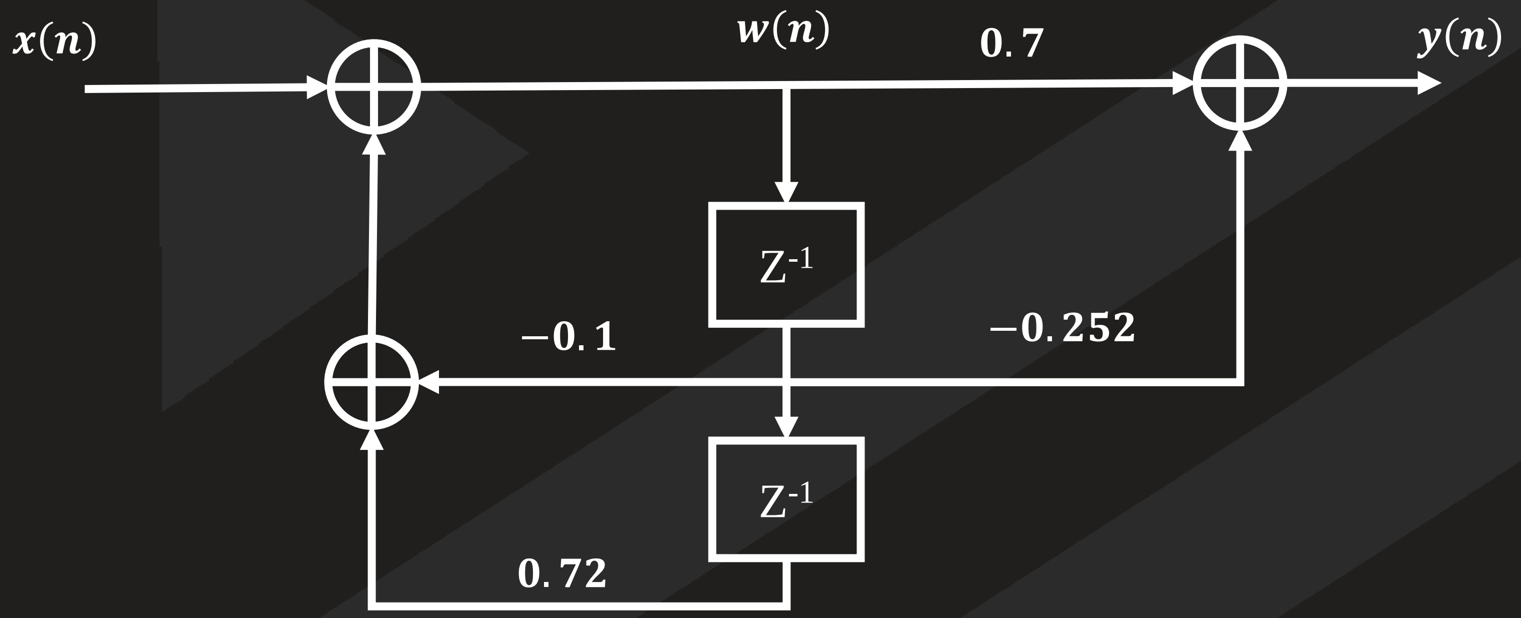
Inverse transform

$$w(n) = x(n) - 0.1w(n-1) + 0.72w(n-2)$$

Direct form II realization

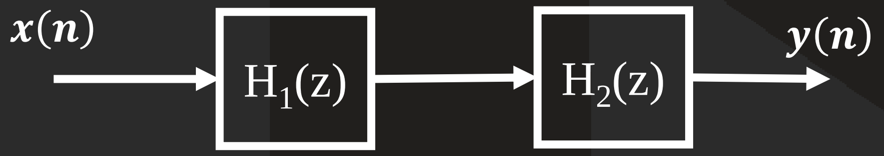
$y(n) = 0.7w(n) - 0.252w(n-1)$

$w(n) = x(n) - 0.1w(n-1) + 0.72w(n-2)$



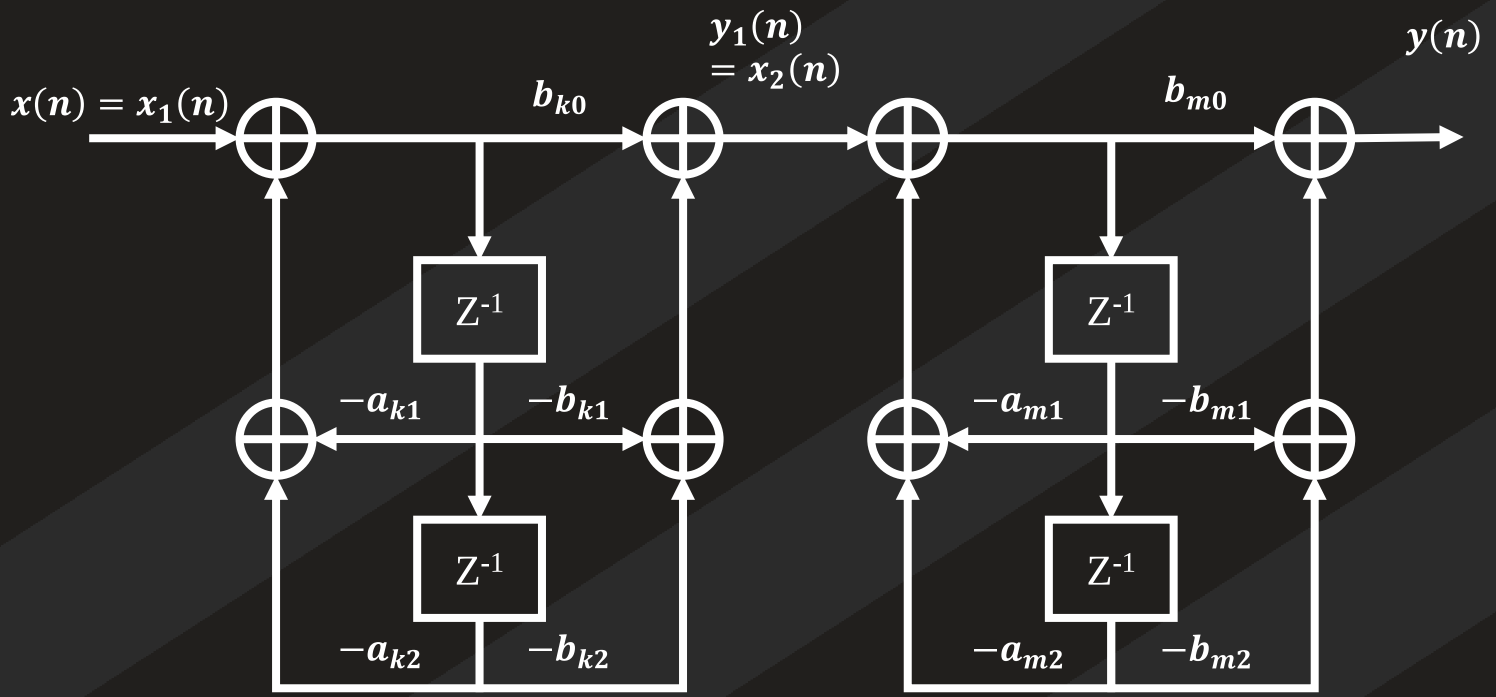
Cascade form realization

Let us consider an IIR system with system function



$$H_1(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

$$H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}$$



Cascade form realization

Q) Realise the system with difference equation

$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$ *in cascade form*

Solution

Taking Z - Transform

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = \frac{3}{4}Y(z)z^{-1} - \frac{1}{8}Y(z)z^{-2} + X(z) + \frac{1}{3}X(z)z^{-1}$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]\left[1 - \frac{1}{4}z^{-1}\right]}$$

$$H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]}$$

$$H_2(z) = \frac{1}{\left[1 - \frac{1}{4}z^{-1}\right]}$$

$$H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]}$$

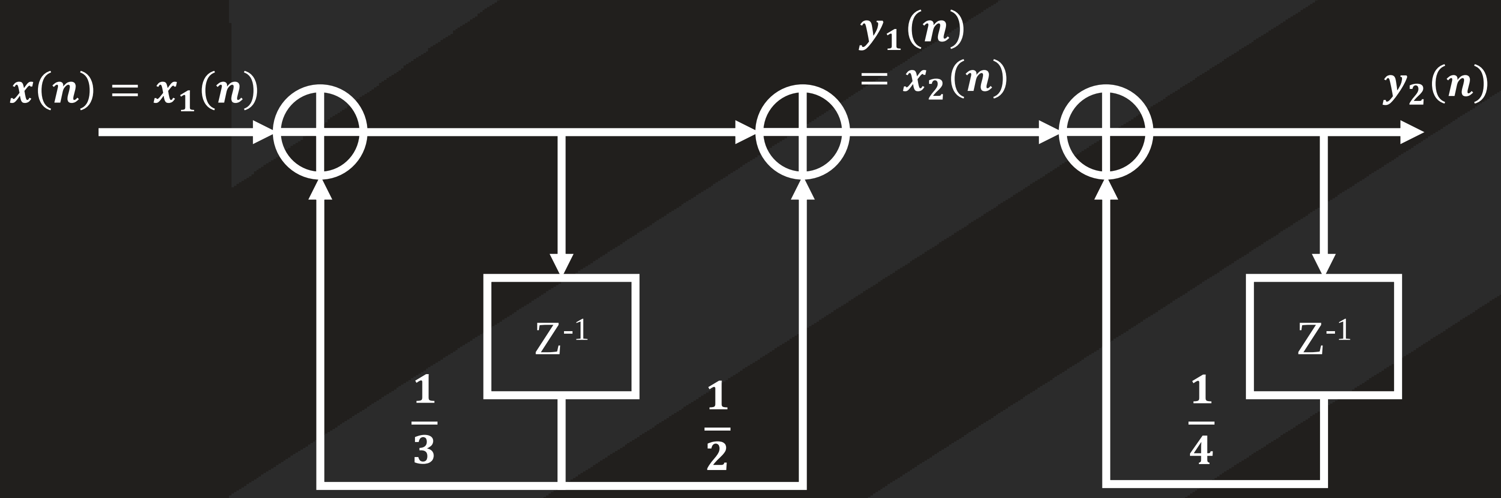
$$H_2(z) = \frac{1}{\left[1 - \frac{1}{4}z^{-1}\right]}$$

$$Y_1(z) = X_1(z) + \frac{1}{3}X_1(z)z^{-1} + \frac{1}{2}Y_1(z)z^{-1}$$

$$y_1(n) = x_1(n) + \frac{1}{3}x_1(n-1) + \frac{1}{2}y_1(n-1)$$

$$Y_2(z) = \mathrm{X}_2(z) + \frac{1}{4}Y_2(z)$$

$$y_2(n) = \mathrm{x}_2(n) + \frac{1}{4}y_2(n-1)$$

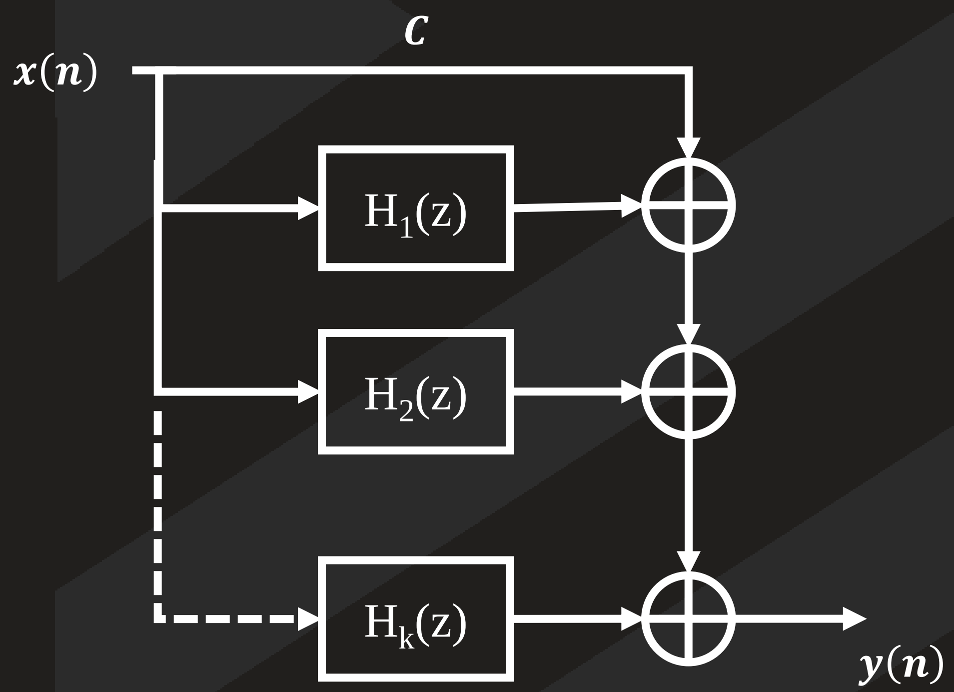


Parallel form realization

A parallel form realisation of an IIR system can be obtained by performing a partial expansion

$$H(z) = C + \sum_{k=1}^{N} \frac{C_k}{1 - P_k z^{-1}} \qquad H(z) = C + \frac{C_1}{1 - P_1 z^{-1}} + \frac{C_2}{1 - P_2 z^{-1}} + \cdots + \frac{C_N}{1 - P_N z^{-1}}$$

$$Y(z) = cX(z) + H_1(z)X(z) + H_2(z)X(z) + \cdots + H_N(z)X(z)$$



Parallel form realization

Q) Realise the system given by difference equation

$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.25x(n-2)$ *in parallel form*

Taking Z - Transform

$$Y(z) = -0.1Y(z)z^{-1} + 0.72Y(z)z^{-2} + 0.7X(z) - 0.25X(z)z^{-2}$$

$$Y(z)[1 + \ 0.1z^{-1} - 0.72z^{-2}] = X(z)[0.7 - 0.25z^{-2}]$$

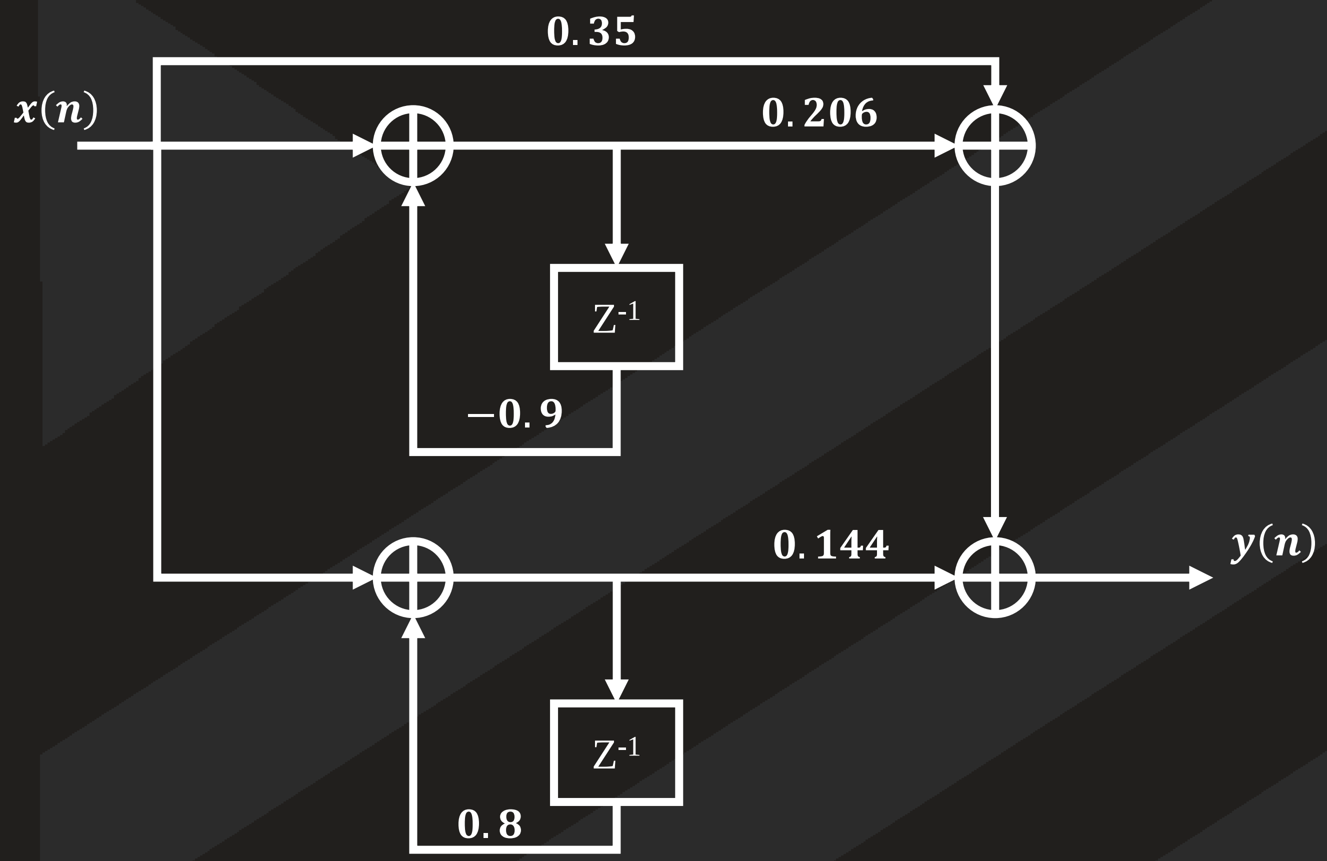
$$H(z) = \frac{0.7 - 0.25z^{-2}}{1 + \ 0.1z^{-1} - 0.72z^{-2}}$$

$$= 0.35 + \frac{-0.035z^{-1} + 0.35}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$= 0.35 + \frac{0.206}{1 + 0.9z^{-1}} + \frac{0.144}{1 - 0.8z^{-1}}$$

Parallel form realization

$$H(z) = 0.35 + \frac{0.206}{1 + 0.9z^{-1}} + \frac{0.144}{1 - 0.8z^{-1}}$$



Multi rate DSP

- The processing of a discrete time signal at different sampling rates in different parts of a system is called multi rate DSP
- Discrete time system that employ sampling rate conversion while processing the discrete time signal are called multi rate DSP system
- The process of converting a signal from one sampling rate to another sampling rate are of two types
  - Down sampling or decimation
  - Up sampling or interpolation

Down sampling

- It is the process of reducing the sampling rate by an integer factor D or M
- Down sampled signal of *x(n)* can be obtained by simply keeping every *M*th sample and removing *(M-1)* in between samples

$$\boldsymbol{x(n)} \longrightarrow \boxed{\downarrow M} \longrightarrow \boldsymbol{y(n) = x(Mn)}$$

$x(n) = \{1, -1, 2, 4, 0, 3, 2, 1, 5\}$ *for M=2*

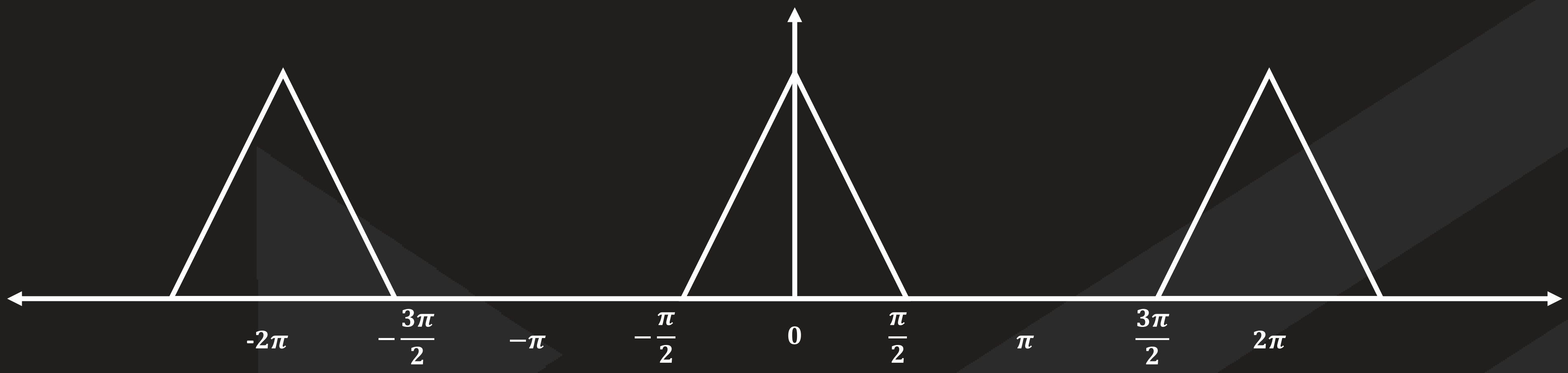
$x(Mn) = \{1, 2, 0, 2, 5\}$

Spectrum of the down sampled signal

$$Y\left(e^{j\omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)$$

- If the Fourier transform of the input-signal of a down samples in $X\left(e^{j\omega}\right)$, then the Fourier transform Y$\left(e^{j\omega}\right)$ of the output signal y(n) is a sum of M uniformly shifted and stretched version of $X\left(e^{j\omega}\right)$ scaled by a factor of 1/M

Q) Consider a spectrum of input signal $X(e^{j\omega})$ with a bandwidth of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ shown , when the signal is down sampled by a factor D, sketch the spectrum of a down sampled signal for sampling rate reduction factor D=2,3



For M/D=2

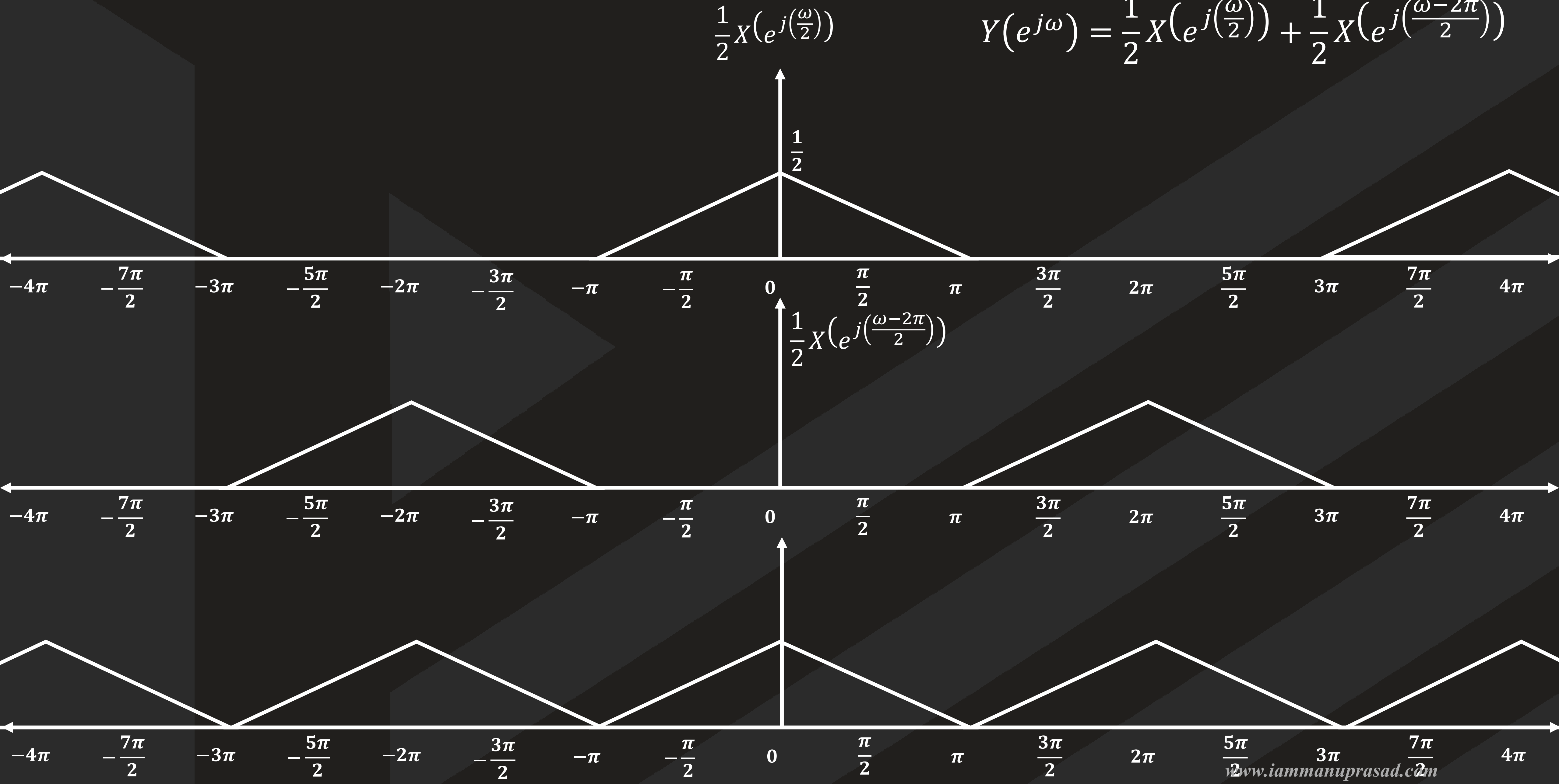
$$Y(e^{j\omega}) = \frac{1}{2}\sum_{k=0}^{1} X\left(e^{j\left(\frac{\omega-2\pi k}{2}\right)}\right)$$

$$Y(e^{j\omega}) = \frac{1}{2}X\left(e^{j\left(\frac{\omega}{2}\right)}\right) + \frac{1}{2}X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)$$

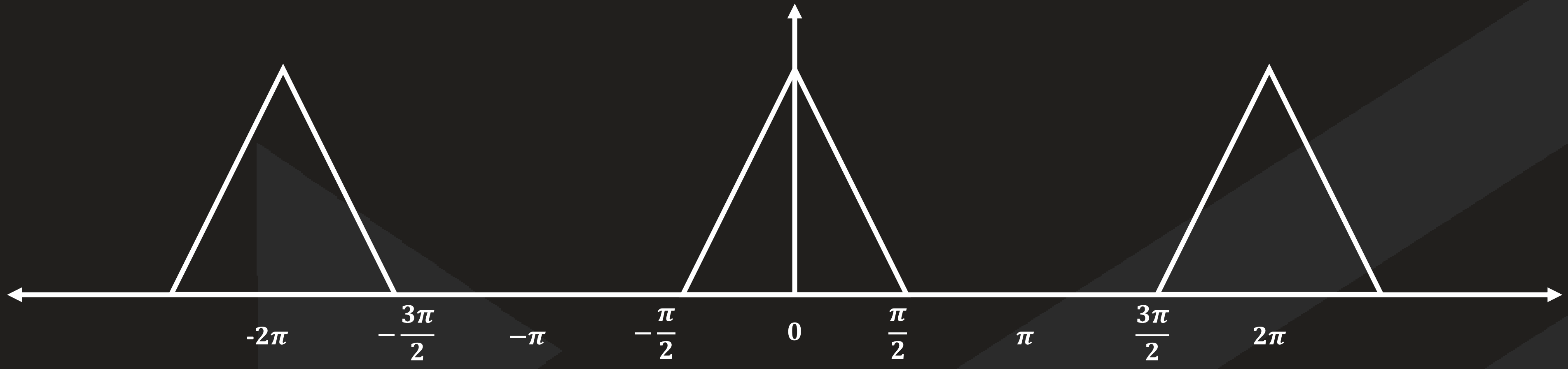
When D=2

Bandwidth =D.BW $=2\pi$

$$Y\left(e^{j\omega}\right) = \frac{1}{2}X\left(e^{j\left(\frac{\omega}{2}\right)}\right) + \frac{1}{2}X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)$$



Q) Consider a spectrum of input signal $X(e^{j\omega})$ with a bandwidth of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ shown , when the signal is down sampled by a factor D, sketch the spectrum of a down sampled signal for sampling rate reduction factor D=2,3



For M/D=3

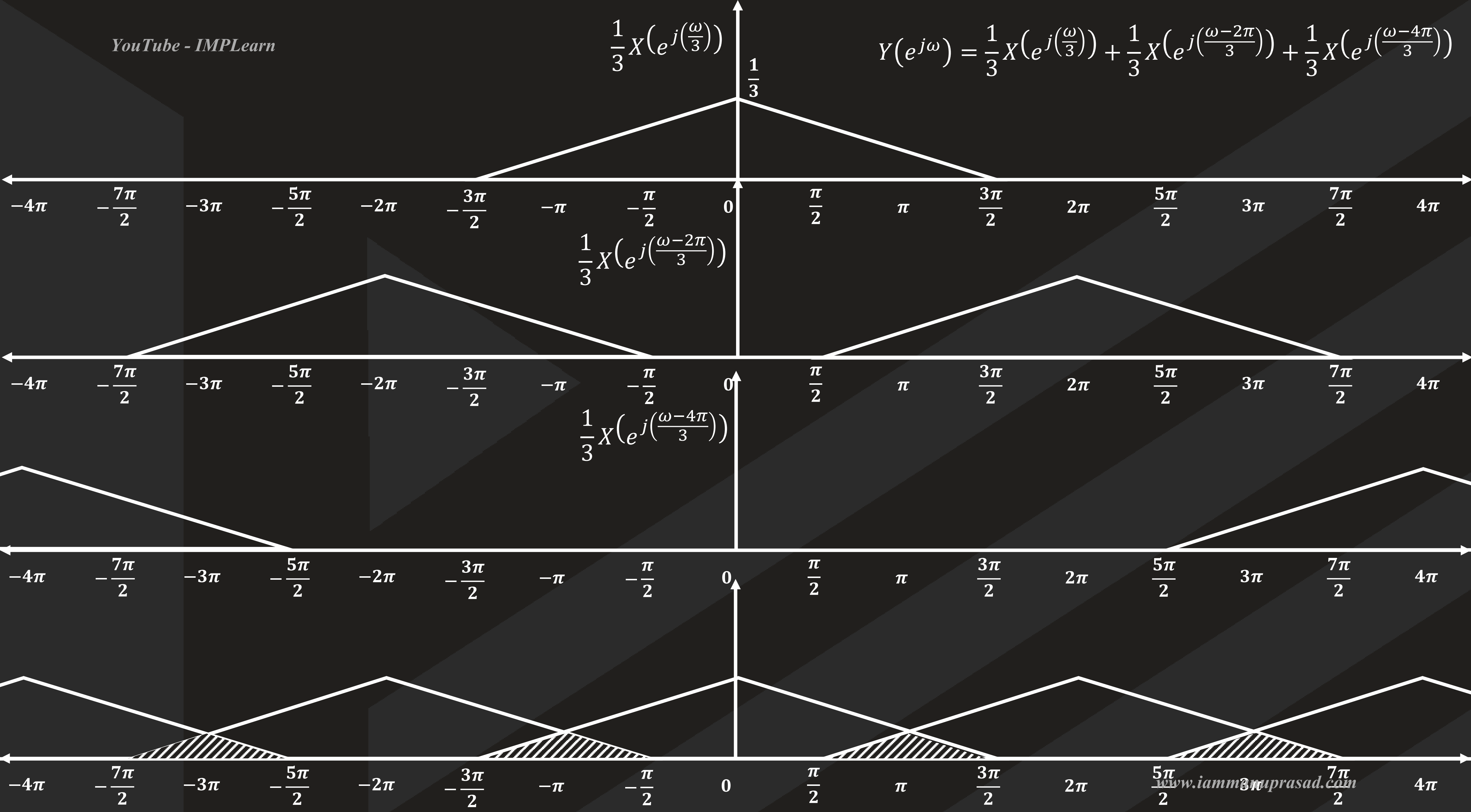
$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega-2\pi k}{3}\right)}\right)$$

$$Y(e^{j\omega}) = \frac{1}{3}X\left(e^{j\left(\frac{\omega}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)$$

When D=3

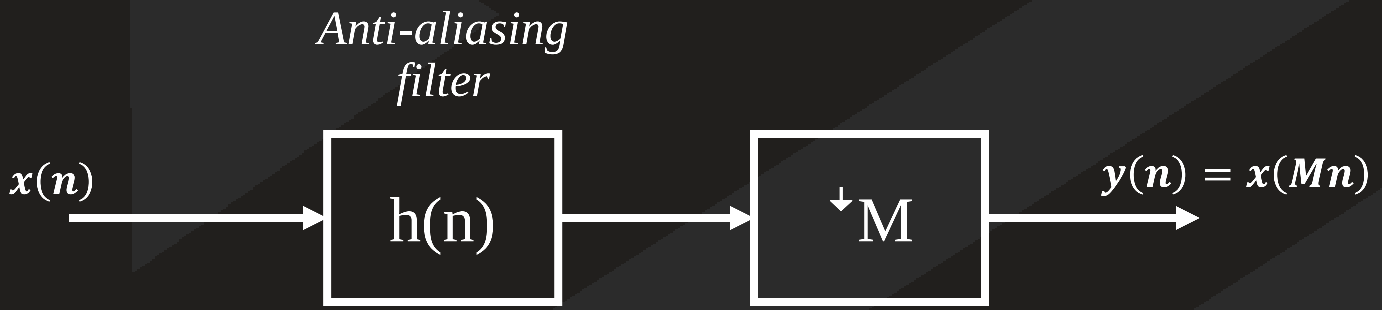
Bandwidth =D.BW $=3\pi$

$$Y(e^{j\omega}) = \frac{1}{3}X\left(e^{j\left(\frac{\omega}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)$$



Anti-aliasing filters

- *In order to avoid aliasing the input signal should be band limited to π/D for decimation by a factor of D*



Up sampling

- *It is the process of increasing the sampling rate by an integer factor I or L*
- *Up sampled signal of x(n) can be obtained by a factor of L by L-1 equally spaced zeros between each pairs of samples*

$$\boldsymbol{x(n)} \longrightarrow \boxed{\uparrow L} \longrightarrow \boldsymbol{y(n) = x\left(\frac{n}{L}\right)}$$

$x(n) = \{1, 2, 3, 4, 5\}$ *for I/L=3*

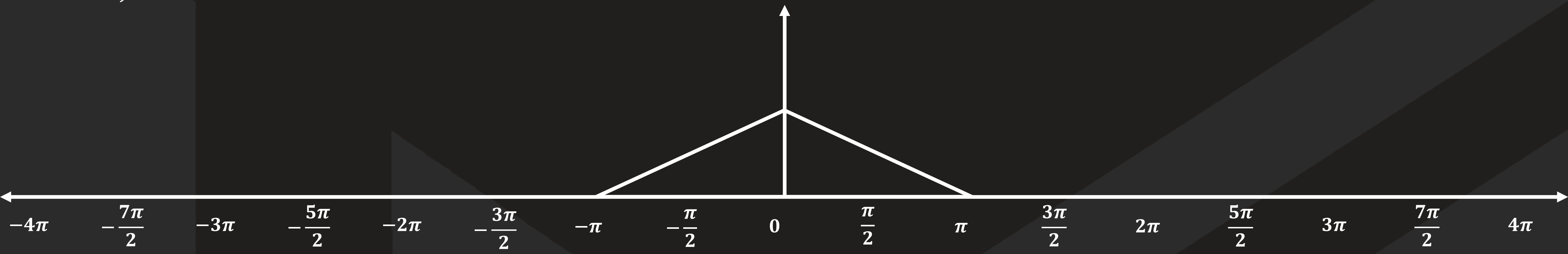
$$x\left(\frac{n}{L}\right) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 4,0,0\ \}$$

Spectrum of the up sampled signal

$$Y(e^{j\omega}) = X(e^{j\omega I})$$

- The term $X(e^{j\omega I})$ is the frequency compressed version of $X(e^{j\omega})$ by a factor I.
- If the frequency response is periodic with 2π , the $X(e^{j\omega I})$ will repeat I times in a period of 0 to 2π in the spectrum of up sampled

Q) The spectrum of discrete time signal is shown below. Draw the spectrum of the signal if it is upscaled by I=2,3

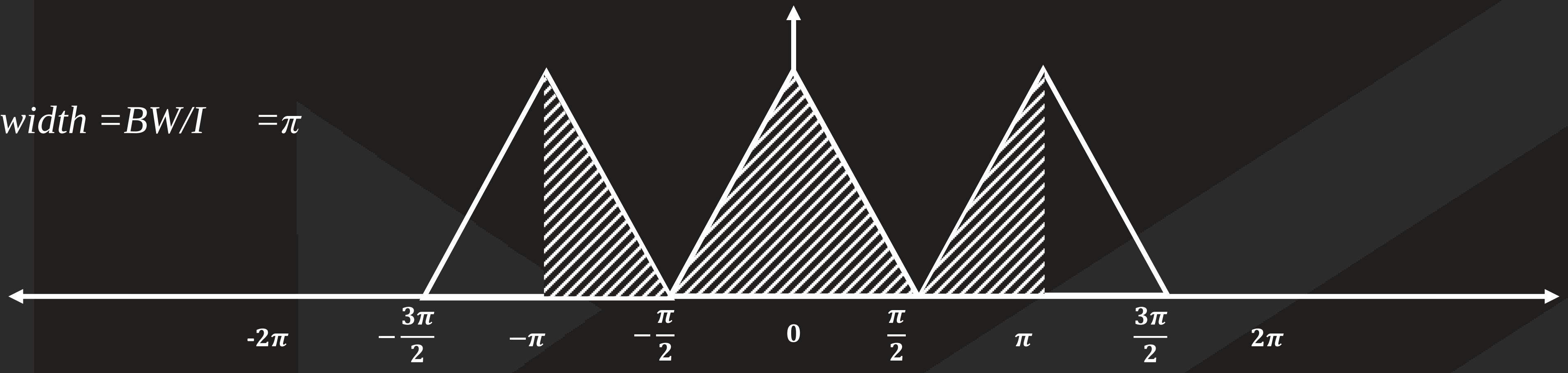


For I or L=2

$$Y\left(e^{j\omega}\right) = X\left(e^{j2\omega}\right)$$

When I=2

Bandwidth =BW/I =π

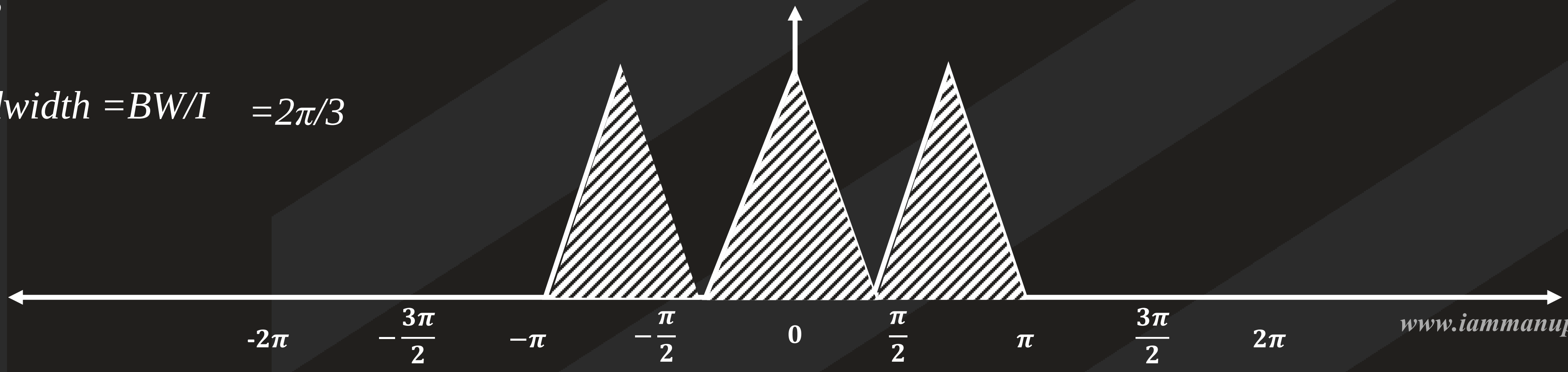


For I or L=3

$$Y\left(e^{j\omega}\right) = X\left(e^{j3\omega}\right)$$

When I=3

Bandwidth =BW/I =2π/3



Anti-imaging filters

- *When up sampled by a factor of I, the output spectrum will have I images in each period with each image bandwidth to* $\frac{\pi}{I}$
- *Since the frequency spectrum in the range to 0 to* $\frac{\pi}{I}$ *are unique and we have to filter the other images*
- *Hence the output of up samples is passed through a lowpass filter with band width* $\frac{\pi}{I}$
- *Since the lowpass filter is designed to avoid multiple images in output spectrum , it also called anti-imaging filter*